Note: General Relativity

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Abstract

Reference: https://www.zhihu.com/question/53496530/answer/544322909

Contents

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1 Gravitational potential

ref: https://www.zhihu.com/question/416630965

$$\vec{F} = \frac{GMm}{r^2}\vec{e_r} = \frac{GMm}{r^3}\vec{r}$$

For gravitational field

$$\vec{A} = \frac{GM}{r^3}\vec{r} = \vec{\nabla}\Phi$$

suppose that gravitational potential Φ is 0 at ∞

$$\Phi = -\int_r^\infty \frac{GM}{r^2}dr = -\frac{GM}{r}$$

For one point with mass M in a closed surface,

$$\Phi = \oint_S \vec{A} \cdot d\vec{S}$$

and suppose such the surface is a sphere

$$\Phi = \oint_S \vec{A} \cdot d\vec{S} = \oint_S \frac{GM}{r^2}dS = \frac{GM}{r^2} \cdot 4\pi r^2 = 4\pi GM$$

we get

$$M = \iiint_V \rho dV = \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \int_0^r r^2 dr \rho = 4\pi\rho \cdot \frac{r^3}{3}$$

From Gauss' law

$$4\pi G \iiint_V \rho dV = \oint_S \vec{A} \cdot d\vec{S} = \iiint_V \vec{\nabla} \cdot \vec{A} dV$$

we obtain Poisson equation for gravitational field:

$$\boxed{\nabla^2\Phi = \vec{\nabla} \cdot \vec{\nabla}\Phi = \vec{\nabla} \cdot \vec{A} = 4\pi G\rho}$$

In addition, we could get the same result by the method below

$$\nabla^2\Phi = \vec{\nabla} \cdot \vec{A} = \vec{\nabla} \cdot (\frac{GM}{r^3}\vec{r}) = \vec{\nabla} \cdot \left(\frac{4}{3}\pi G\rho(\vec{x} + \vec{y} + \vec{z})\right) = \frac{4}{3}\pi G\rho\vec{\nabla} \cdot (\vec{x} + \vec{y} + \vec{z}) = 4\pi G\rho$$

2 Gauge & Tensor

We select the gauge below for Minkowski spacetime $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu}x^\mu x^\nu$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Poincaré gauge: $ds^2 = (dx^2 + dy^2)/y^2$

electromagnetic tensor
https://www.zhihu.com/question/383212450/answer/1139032723
https://en.wikipedia.org/wiki/Electromagnetic_tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

$$F_{\mu\nu} = \eta_{\mu\alpha} F^{\alpha\beta} \eta_{\beta\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

The energy–momentum tensor is symmetric, $T^{ab} = T^{ba}$, in which, T^{00} is energy density; $T^{0i} = T^{i0}$ refers to the flux of relativistic mass across the x^i surface is equivalent to the density of the ith component of linear momentum; T^{ij} represent flux of ith component of linear momentum across the x^j surface. In particular, T^{ii} refers to normal stress, and T^{ij} refers to shear stress.

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \qquad \text{(perfect fluid)}$$

3 Action principle

Lagrangian

$$L = T - V$$

Hamiltonian

$$H = T + V$$

The action for the system with potential is defined as

$$S = \int L dt = \int dt \left[\frac{1}{2} m \left(\frac{d\vec{x}}{dt} \right)^2 - V(x) \right]$$

If the potential is zero, then

$$S = \int L dt = \int dt \left[\frac{1}{2} m \left(\frac{d\vec{x}}{dt} \right)^2 \right] = \frac{1}{2} m \int \frac{(d\vec{x})^2}{dt}$$

Ignore the high orders in the below equation,

$$\left(a - \frac{\epsilon^2}{2a} \right)^2 = a^2 - \epsilon^2 + \frac{\epsilon^4}{4a^2}$$

we have the approximation

$$\sqrt{a^2 - \epsilon^2} \approx a - \frac{\epsilon^2}{2a} + \cdots$$

$$\frac{\epsilon^2}{2a} \approx -\sqrt{a^2-\epsilon^2} + a + \cdots$$

therefore

$$\frac{(\Delta\vec{x})^2}{2\Delta t} = c\frac{(\Delta\vec{x})^2}{2c\Delta t} = -c\sqrt{(c\Delta t)^2 - (\Delta\vec{x})^2} + c^2\Delta t$$

namely

$$S = -mc\int \sqrt{(cdt)^2 - (d\vec{x})^2} + mc^2 \int dt$$

the last term only depends on $t_{initial}$ and t_{final}, distributing nothing toward the integration.

$$\boxed{S = -m\int \sqrt{dt^2 - (d\vec{x})^2} = -m\int \sqrt{-\eta_{\mu\nu}dX^\mu dX^\nu}}$$

$$S = -m\int \sqrt{-\eta_{\mu\nu}dX^\mu dX^\nu} = -m\int d\zeta\sqrt{-\eta_{\mu\nu}\frac{dX^\mu}{d\zeta}\frac{dX^\nu}{d\zeta}} = \int L d\zeta$$

$$L = -m\sqrt{-\eta_{\mu\nu}\frac{dX^\mu}{d\zeta}\frac{dX^\nu}{d\zeta}}$$

For massless particles, re-define

$$\tilde{S} = -\frac{1}{2}\int d\zeta\left(\sigma(\zeta)\left(\frac{dX}{d\zeta}\right)^2 + \frac{m^2}{\sigma(\zeta)}\right)$$

where

$$\left(\frac{dX}{d\zeta}\right)^2 = -\eta_{\mu\nu}\frac{dX^\mu}{d\zeta}\frac{dX^\nu}{d\zeta}$$

Since $\frac{d\sigma}{d\zeta}$ does not appear, we apply Euler-Lagrange equation,

$$\frac{d}{d\zeta}\left(\frac{\delta\tilde{S}}{\delta\frac{d\sigma}{d\zeta}}\right) - \frac{\delta\tilde{S}}{\delta\sigma} = 0 \qquad \Rightarrow \qquad \frac{\delta\tilde{S}}{\delta\sigma} = 0$$

obtaining

$$\frac{m^2}{\sigma(\zeta)^2} = \left(\frac{dX}{d\zeta}\right)^2$$

Insert X^λ into E-L eq. Since X^λ is not the manifest variable of $\tilde{S}$

$$\frac{d}{d\zeta}\left(\frac{\delta\tilde{S}}{\delta\frac{dX^\lambda}{d\zeta}}\right) = \frac{d}{d\zeta}\left(\sigma\eta_{\mu\lambda}\frac{dX^\mu}{d\zeta}\right) = \frac{d}{d\zeta}\left(\frac{m}{\sqrt{(\frac{dX}{d\zeta})^2}}\eta_{\mu\lambda}\frac{dX^\mu}{d\zeta}\right) = 0$$

With the definition $(\frac{dX}{d\tau})^2 = 1$, we replace $d\zeta$ as $d\tau$. Therefore, $\frac{d^2X^\mu}{d\tau^2} = 0$, the same form of S appears again.

The action for massless particles

$$S_{massless} = \frac{1}{2}\int d\zeta\left(\sigma\eta_{\mu\nu}\frac{dX^\mu}{d\zeta}\frac{dX}{d\zeta}\right)$$

Potential outside the square root, Option E : (electromagnetism)

$$S = -\int \left[m\sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu} + V(x)dt \right]$$

Potential inside the square root, Option G : (gravity)

$$S = -m \int \sqrt{\left(1 + \frac{2V}{m}\right) dt^2 - d\vec{x}^2}$$

Option E improved :

$$S = -\int \left[m\sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu} + A_\mu(x)dx^\mu \right]$$

Option G improved :

$$S = -m \int \sqrt{-g_{\mu\nu}(x)dx^\mu dx^\nu}$$

4 Geodesic

Geodesic equation

$$\frac{\partial^2 X^\lambda}{\partial \tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau} = 0$$

with Christoffel symbol $\Gamma^\lambda_{\mu\nu}$

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\frac{\partial g_{\mu\sigma}}{\partial X^\nu} + \frac{\partial g_{\nu\sigma}}{\partial X^\mu} - \frac{\partial g_{\mu\nu}}{\partial X^\sigma})$$

Suppose $V^\mu = \frac{dX^\mu}{dl}$, then we get the equation

$$\frac{dV^\rho}{dl} + \Gamma^\rho_{\mu\nu} V^\mu V^\nu = 0$$

Since there would be external forces (without gravity as the essential property of spacetime), we write geodesic eq as

$$\frac{\partial^2 X^\lambda}{\partial \tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau} = f^\lambda(X)$$

As for the gauge ($d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$)

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2 = -dt^2 + a(t)^2(dr^2 + r^2 d\Omega^2)$$

the action

$$S = -m \int d\tau \left[\left(\frac{dt}{d\tau}\right)^2 - a(t)^2 \left(\frac{d\vec{x}}{d\tau}\right)^2 \right]^{\frac{1}{2}}$$

insert that into geodesic eq.

$$\frac{d^2}{d\tau^2} + a(t)\dot{a}(t) \left(\frac{d\vec{x}}{d\tau}\right)^2 = 0$$

$$\frac{d}{d\tau}\left(a(t)^2\frac{d\vec{x}}{d\tau}\right)=0 \qquad \Rightarrow \qquad \frac{d^2\vec{x}}{d\tau^2}+\frac{2\dot{a}(t)}{a(t)}\frac{dt}{d\tau}\frac{d\vec{x}}{d\tau}=0$$

$$\left(\frac{dt}{d\tau}\right)^2-a(t)^2\left(\frac{d\vec{x}}{d\tau}\right)^2=1$$

$$\Gamma^0_{ij}=a\dot{a}\delta_{ij} \qquad \Gamma^i_{0j}=\Gamma^i_{j0}=\frac{\dot{a}}{a}\delta^i_j$$

Introduce the gauge related to the universe $a(t) = e^{Ht}$, where H is Hubble's constant. Suppose the moment to give out signals as t_S, then $R_{max} = e^{-t_S}$. If $R > R_{max}$, signals would not arrive at R; if $R_{max} > R > \frac{1}{2}R_{max}$, signals could arrive but no replies received.

Ref - https://zhuanlan.zhihu.com/p/163704300

For the sake of easier calculations, we multiply $dx^\mu dx^\nu$ with Christoffel symbol

$$\begin{aligned}\Gamma^\lambda_{\mu\nu}dx^\mu dx^\nu &= \frac{1}{2}g^{\lambda\sigma}(\frac{\partial g_{\mu\sigma}}{\partial x^\nu}+\frac{\partial g_{\nu\sigma}}{\partial x^\mu}-\frac{\partial g_{\mu\nu}}{\partial x^\sigma})dx^\mu dx^\nu \\ &= \frac{1}{2}g^{\lambda\sigma}(dg_{\mu\sigma}dx^\mu+dg_{\nu\sigma}dx^\nu-\frac{\partial g_{\mu\nu}}{\partial x^\sigma}dx^\mu dx^\nu) \\ &= g^{\lambda\sigma}(dg_{\mu\sigma}dx^\mu-\frac{1}{2}\frac{\partial ds^2}{\partial x^\sigma})\end{aligned}$$

Example 1, for $ds^2 = r^2d\theta^2 + r^2\sin^2\theta d\phi^2 = 0dt^2 + 0dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2$

$$g_{\mu\nu}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&r^2&0\\0&0&0&r^2\sin^2\theta\end{pmatrix} \qquad g^{\mu\nu}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1/r^2&0\\0&0&0&1/r^2\sin^2\theta\end{pmatrix}$$

$$\begin{aligned}\Gamma^\theta_{\mu\nu}dx^\mu dx^\nu &= g^{\theta\theta}(dg_{\theta\theta}d\theta-\frac{1}{2}\frac{\partial ds^2}{\partial\theta}) \\ &= \frac{1}{r^2}(2rdrd\theta-\frac{1}{2}2r^2\sin\theta\cos\theta d\phi^2) \\ &= \frac{2}{r}drd\theta-\sin\theta\cos\theta d\phi^2\end{aligned}$$

thus

$$\Gamma^\theta_{r\theta}=\Gamma^\theta_{\theta r}=\frac{1}{r} \qquad \Gamma^\theta_{\phi\phi}=-\sin\theta\cos\theta$$

$$\begin{aligned}\Gamma^\phi_{\mu\nu}dx^\mu dx^\nu &= g^{\phi\phi}(dg_{\phi\phi}d\phi-\frac{1}{2}\frac{\partial ds^2}{\partial\phi}) \\ &= \frac{1}{r^2\sin^2\theta}(2r^2\sin\theta\cos\theta d\theta d\phi+2r\sin^2\theta drd\phi-0) \\ &= 2\frac{\cos\theta}{\sin\theta}d\theta d\phi+\frac{2}{r}drd\phi\end{aligned}$$

thus

$$\Gamma^\phi_{r\phi}=\Gamma^\phi_{\phi r}=\frac{1}{r} \qquad \Gamma^\phi_{\theta\phi}=\Gamma^\phi_{\phi\theta}=\cot\theta$$

Verify:

$$\begin{aligned}\Gamma^{\theta}_{r\theta} &= \frac{1}{2}g^{\theta\theta}(\frac{\partial g_{\theta\theta}}{\partial r} + \frac{\partial g_{\theta\theta}}{\partial \theta} - \frac{\partial g_{r\theta}}{\partial \theta}) \\ &= \frac{1}{2}g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial r} \\ &= \frac{1}{2}\cdot\frac{1}{r^2}\cdot 2r = \frac{1}{r}\end{aligned}$$

$$\begin{aligned}\Gamma^{\phi}_{r\phi} &= \frac{1}{2}g^{\phi\phi}(\frac{\partial g_{\phi\phi}}{\partial r} + \frac{\partial g_{\phi r}}{\partial \phi} - \frac{\partial g_{r\phi}}{\partial \phi}) \\ &= \frac{1}{2}g^{\phi\phi}\frac{\partial g_{\phi\phi}}{\partial r} \\ &= \frac{1}{2}\cdot\frac{1}{r^2\sin^2\theta}\cdot 2r\sin^2\theta = \frac{1}{r}\end{aligned}$$

Consider $ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$

$$\begin{aligned}\Gamma^{r}_{\mu\nu}dx^{\mu}dx^{\nu} &= g^{rr}\left(dg_{rr}dr - \frac{1}{2}\frac{\partial ds^2}{\partial r}\right) \\ &= -\frac{1}{2}(2rd\theta^2 + 2r\sin^2\theta d\phi^2) \\ &= -rd\theta^2 - r\sin^2\theta d\phi^2\end{aligned}$$

$$\Gamma^{r}_{\theta\theta} = -r \qquad \Gamma^{r}_{\phi\phi} = -r\sin^2\theta$$

Example 2, $ds^2 = -N^2(t)dt^2 + a^2(t)\hat{g}_{ij}dx^i dx^j$

$$\begin{aligned}\Gamma^{t}_{\mu\nu}dx^{\mu}dx^{\nu} &= g^{tt}(dg_{tt}dt - \frac{1}{2}\frac{\partial ds^2}{\partial t}) \\ &= \frac{1}{-N^2}\left(-2N\dot{N}dt^2 - \frac{1}{2}(-2N\dot{N}dt^2 + 2a\dot{a}d\hat{s}^2)\right) \\ &= \frac{1}{-N^2}(-N\dot{N}dt^2 - a\dot{a}d\hat{s}^2) \\ &= \frac{\dot{N}}{N}dt^2 + \frac{a\dot{a}}{N^2}d\hat{s}^2\end{aligned}$$

$$\Gamma^{t}_{tt} = \frac{\dot{N}}{N} \qquad \Gamma^{t}_{ij} = \frac{a\dot{a}}{N^2}\hat{g}_{ij}$$

$$\begin{aligned}\Gamma^{i}_{\mu\nu}dx^{\mu}dx^{\nu} &= \frac{\hat{g}^{ij}}{a^2}\left(d(a^2\hat{g}_{jk})dx^k - \frac{1}{2}\frac{\partial ds^2}{\partial x^j}\right) \\ &= \frac{\hat{g}^{ij}}{a^2}\left(2a\dot{a}\hat{g}_{jk}dtdx^k + a^2 d\hat{g}_{jk}dx^k - \frac{a^2}{2}\frac{\partial d\hat{s}^2}{\partial x^j}\right) \\ &= \frac{2\dot{a}}{a}dtdx^i + \hat{\Gamma}^i\end{aligned}$$

$$\Gamma^{i}_{tj} = \Gamma^{i}_{jt} = \frac{\dot{a}}{a}\delta^i_j \qquad \Gamma^{i}_{jk} = \hat{\Gamma}^{i}_{jk}$$

Example 3 $ds^2 = I_{ab}(u)du^a du^b + r^2(u)\hat{g}_{ij}(x)dx^i dx^j$

$$\begin{aligned}\Gamma^{a} &= I^{ab}\left(dI_{bc}du^c - \frac{1}{2}\frac{\partial d\bar{s}^2}{\partial u^b} - \frac{1}{2}\frac{\partial r^2}{\partial u^b}d\hat{s}^2\right) \\ &= \bar{\Gamma}^a - rI^{ab}\partial_b r d\hat{s}^2\end{aligned}$$

$$\Gamma^{a}_{bc} = \bar{\Gamma}^{a}_{bc} \qquad \Gamma^{a}_{ij} = -rI^{ab}\partial_b r\hat{g}_{ij}$$

$$\Gamma^i = \frac{\hat{g}^{ij}}{r^2}\left(d(r^2\hat{g}_{jk})dx^k - \frac{1}{2}\frac{\partial ds^2}{\partial x^j}\right)$$
$$= \frac{\hat{g}^{ij}}{r^2}\left(2r(\partial_a r)\hat{g}_{jk}du^a dx^k + r^2 d\hat{g}_{jk}dx^k - \frac{r^2}{2}\frac{\partial d\hat{s}^2}{\partial x^j}\right)$$
$$= 2\frac{\partial_a r}{r}du^a dx^i + \hat{\Gamma}^i$$
$$\Gamma^i_{aj} = \Gamma^i_{ja} = \frac{\partial_a r}{r}\delta^i_j \qquad \Gamma^i_{jk} = \hat{\Gamma}^i_{jk}$$

How to derive $ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$?

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} x^2 = r^2\sin^2\theta\cos^2\phi \\ y^2 = r^2\sin^2\theta\sin^2\phi \\ z^2 = r^2\cos^2\theta \end{cases}$$

$$\begin{cases} dx^2 = (\sin^2\theta\cos^2\phi)dr^2 + (r^2\cos^2\phi)d\sin^2\theta + (r^2\sin^2\theta)d\cos^2\phi \\ dy^2 = (\sin^2\theta\sin^2\phi)dr^2 + (r^2\sin^2\phi)d\sin^2\theta + (r^2\sin^2\theta)d\sin^2\phi \\ dz^2 = (\cos^2\theta)dr^2 + (r^2)d\cos^2\theta \end{cases}$$

$$ds^2 = dx^2 + dy^2 + dz^2 = \begin{cases} (\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta)dr^2 \\ r^2\left[(\cos^2\phi + \sin^2\phi)d\sin^2\theta + d\cos^2\theta\right] \\ r^2\sin^2\theta(d\cos^2\phi + d\sin^2\phi) \end{cases}$$

5 Notions & Curvature tensor

Define covariant derivative:

$$D_\lambda W^\mu \equiv \partial_\lambda W^\mu + \Gamma^\mu_{\lambda\nu}W^\nu$$

Introduce notions

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \qquad W^\mu_{,\lambda} \equiv \partial_\lambda W^\mu \qquad W^\mu_{;\lambda} \equiv D_\lambda W^\mu$$
$$W^\mu_{;\lambda} = W^\mu_{,\lambda} + \Gamma^\mu_{\lambda\nu}W^\nu$$

A^μ and B^ν are vector fields. Define commutator $C = [A, B]$:

$$C^\nu = [A,B]^\nu = A^\mu(\partial_\mu B^\nu) - B^\mu(\partial_\mu A^\nu) = A^\mu(D_\mu B^\nu) - B^\mu(D_\mu A^\nu)$$

$$D_\mu W^\mu = \partial_\mu W^\mu + \left(\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}\right)W^\mu$$
$$D_\lambda W^\mu \equiv \partial_\lambda W^\mu + \Gamma^\mu_{\lambda\nu}W^\nu$$

Riemann curvature tensor

$$R^\sigma_{\rho\mu\nu} = (\partial_\mu\Gamma^\sigma_{\nu\rho} + \Gamma^\sigma_{\mu\kappa}\Gamma^\kappa_{\nu\rho}) - (\partial_\nu\Gamma^\sigma_{\mu\rho} + \Gamma^\sigma_{\nu\kappa}\Gamma^\kappa_{\mu\rho})$$

Note that this tensor anti-symmetric(https://zhuanlan.zhihu.com/p/163705623)

$$R^\lambda_{\rho\mu\nu} = \left(\partial_\mu\Gamma^\lambda_{\nu\rho} - \partial_\nu\Gamma^\lambda_{\mu\rho}\right) + \left(\Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho}\right) = -R^\lambda_{\rho\nu\mu}$$

Outer product and Tensor product

$$dx^\mu \wedge dx^\nu = dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu$$

$$\begin{aligned} R^\lambda_{\rho\mu\nu} dx^\mu \otimes dx^\nu &= \frac{1}{2!}\left(R^\lambda_{\rho\mu\nu} dx^\mu \otimes dx^\nu + R^\lambda_{\rho\nu\mu} dx^\nu \otimes dx^\mu\right) \\ &= \boxed{\frac{1}{2!} R^\lambda_{\rho\mu\nu} dx^\mu \wedge dx^\nu = \Omega^\lambda_\rho} \end{aligned}$$

$$d(f dx^\mu \wedge dx^\nu \wedge \cdots) = df \wedge dx^\mu \wedge dx^\nu \wedge \cdots = (\partial_\lambda f) dx^\lambda \wedge dx^\mu \wedge dx^\nu \wedge \cdots$$

In order to calculate curvature tensors faster, attempt:

$$\begin{aligned} \frac{1}{2}\left(\partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho}\right) dx^\mu \wedge dx^\nu &= \left(\partial_\mu \Gamma^\lambda_{\nu\rho} dx^\mu \wedge dx^\nu - \partial_\nu \Gamma^\lambda_{\mu\rho} dx^\mu \wedge dx^\nu\right) \\ &= \left(\partial_\mu \Gamma^\lambda_{\nu\rho} dx^\mu \wedge dx^\nu - \partial_\mu \Gamma^\lambda_{\nu\rho} dx^\nu \wedge dx^\mu\right) \\ &= \partial_\mu \Gamma^\lambda_{\nu\rho} dx^\mu \wedge dx^\nu \\ &= \left(d\Gamma^\lambda_{\nu\rho} \wedge dx^\nu\right) \end{aligned}$$

$$\begin{aligned} \frac{1}{2}\left(\Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho}\right) dx^\mu \wedge dx^\nu =& \frac{1}{2}\left(\Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} dx^\mu \wedge dx^\nu - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho} dx^\mu \wedge dx^\nu\right) \\ =& \frac{1}{2}\left(\Gamma^\lambda_{\mu\sigma} dx^\mu \wedge \Gamma^\sigma_{\nu\rho} dx^\nu + \Gamma^\lambda_{\nu\sigma} dx^\nu \wedge \Gamma^\sigma_{\mu\rho} dx^\mu\right) \\ =& \left(\Gamma^\lambda_{\mu\sigma} dx^\mu\right) \wedge \left(\Gamma^\sigma_{\nu\rho} dx^\nu\right) = A^\lambda_\sigma \wedge A^\sigma_\rho \end{aligned}$$

$$\begin{aligned} R^\lambda_{\rho\mu\nu} dx^\mu \otimes dx^\nu &= \frac{1}{2!} R^\lambda_{\rho\mu\nu} dx^\mu \wedge dx^\nu = \Omega^\lambda_\rho \\ &= \frac{1}{2}\left[\left(\partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho}\right) + \left(\Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho}\right)\right] dx^\mu \wedge dx^\nu \\ &= \frac{1}{2}\left(\partial_\mu \Gamma^\lambda_{\nu\rho} - \partial_\nu \Gamma^\lambda_{\mu\rho}\right) dx^\mu \wedge dx^\nu + \frac{1}{2}\left(\Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\rho} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\rho}\right) dx^\mu \wedge dx^\nu \\ &= \left(d\Gamma^\lambda_{\nu\rho} \wedge dx^\nu\right) + \left(\Gamma^\lambda_{\mu\sigma} dx^\mu\right) \wedge \left(\Gamma^\sigma_{\nu\rho} dx^\nu\right) \\ &= \boxed{dA^\lambda_\rho + A^\lambda_\sigma \wedge A^\sigma_\rho = \Omega^\lambda_\rho} \end{aligned}$$

$$R_{klij} = g_{lm} R^m_{kij}$$

$$R = R_{klij} dx^k \otimes dx^l \otimes dx^i \otimes dx^j$$

$$R = R^l_{kij} dx^k \otimes \frac{\partial}{\partial x^l} \otimes dx^i \otimes dx^j$$

Introduce Bianchi identities

$$R_{abcd} + R_{adbc} + R_{acdb} = R_{a[bcd]} = 0$$

$$\nabla_e R_{abcd} + \nabla_d R_{abec} + \nabla_c R_{abde} = R_{ab[cd;e]} = 0$$

Ricci tensor, or Ricci curvature tensor assigns to each tangent space of the manifold a symmetric bilinear form, describing the volume inside one given Riemannian metric compared to that in n-dim Euclidean space.

$$\mathrm{Ric} = \sum_{ij} R_{ij} dx^i \otimes dx^j = R_{ij} dx^i \otimes dx^j = \sum_k R^k_{ikj} dx^i \otimes dx^j$$

$$R_{\beta\nu} \equiv R^{\alpha}_{\beta\alpha\nu}$$

Moreover, there is curvature scalar:

$$R \equiv g^{\mu\nu} R_{\mu\nu}$$

6 How to derive Einstein field equation

ref - https://www.zhihu.com/question/53496530/answer/544322909

ref - https://www.zhihu.com/question/53496530/answer/258731044

The field equation to describe General Relativity must be such a form: $G^{\mu\nu} = \kappa T^{\mu\nu}$, where $G^{\mu\nu}$ is Einstein tensor, describing how spacetime bends; κ is a coefficient; $T^{\mu\nu}$ represents stress-energy tensor, for the distribution of energies. The left side is the paramount part...

Let's guess. Note that we have some conditions,

- First, there should be a tensor eq, keeping invariant under coordinate transformations;
- According to the symmetry of stress-energy tensor, namely $\nabla_\mu T^{\mu\nu} = 0$, thus $\nabla_\mu G^{\mu\nu} = 0$ for Einstein tensor;
- In classical limit, it should be approximated as Newtonian gravity $\nabla^2 \Phi = 4\pi G\rho$.

In classical mechanics, the gravity is written as $\nabla^2 \Phi = 4\pi G\rho$, where Φ is gravitational potential. In relativistic world, density would be replace by energy-momentum tensor, and Φ turns to the metric $g_{\mu\nu}$.

Therefore, we could guess that the eq should satisfy properties below

$$F(g, \partial_\lambda g, \partial_\gamma \partial_\sigma g)_{\mu\nu} \propto T_{\mu\nu}$$

where $F_{\mu\nu}$ is a (0,2) tensor depending on the gauge and its first/second derivatives. Thus, we try Ricci tensor naturally

$$F_{\mu\nu} = R_{\mu\nu}$$

However, it seems impossible, since we have

$$\nabla^\mu T_{\mu\nu} = 0 \qquad \nabla^\mu R_{\mu\nu} = \frac{1}{2} \nabla_\nu R$$

we then try to construct new term

$$\nabla^\mu (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) = \nabla^\mu R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \nabla^\mu R = \nabla^\mu R_{\mu\nu} - \frac{1}{2} \nabla_\nu R = 0 = \nabla^\mu T_{\mu\nu}$$

then we write

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = G_{\mu\nu} = \kappa T_{\mu\nu}$$

Then next target is the parameter κ. Note the stress-energy tensor for ideal fluids in the classical limit

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}$$

we ignore the pressure for the low velocity

$$T_{\mu\nu} = \rho U_\mu U_\nu$$

in the rest frame of reference, we have

$$T_{00} = \rho$$

Thus,

$$T = g^{\mu\nu} T_{\mu\nu} = -\rho$$

$$R = \kappa\rho$$

as a result, $R_{00} = \frac{1}{2}\kappa\rho$. According to the definition of $R_{\mu\nu}$, we know $R_{00} = R^{\mu}_{0\mu 0}$. Since Riemannian tensor is anti-symmetric, $R^{0}_{000} = 0$. In addition, we suppose the gravitation field is rest under Newtonian limit, and all first derivatives of the gauge equal to zero.

$$R_{00} = -\frac{1}{2}\nabla^2 h_{00}$$

where h_{00} is the perturbation of the gauge, $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, then

$$\nabla^2 h_{00} = -\kappa\rho$$

From the geodesic equation, $h_{00} = -2\Phi$

$$\nabla^2(-2\Phi) = -2\nabla^2\Phi = -\kappa\rho$$

Compared with $\nabla^2\Phi = 4\pi G\rho$ in classical mechanics, we obtain immediately

$$\kappa = 8\pi G$$

then we determine Einstein Field Equations:

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Furthermore, $\nabla^\mu g_{\mu\nu} = 0$. If we add one term $\Lambda g_{\mu\nu}$ on the left, the eq still satisfies, since all covariant derivatives equal to 0.

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

where Λ is called cosmological constant, regarded as a possible form of Dark energy. To obtain that, we could add one term on the right for vacuum energy density, $T^{(vac)}_{\mu\nu} = \rho^{(vac)} g_{\mu\nu}$, furnishing

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi G(T_{\mu\nu} + T^{(vac)}_{\mu\nu})$$

simply $\rho^{(vac)} = \frac{\Lambda}{8\pi G}$. Since vacuum energy density is only different from the cosmological constant with one constant, we often regard these two conceptions equivalently.

7 Experimental evidences of GR

In 1927, G. D. Birkhoff proved, the gauge with spherical symmetry in vacuum must be rest, written as one form independent with time t.

$$ds^2 = b(r)c^2dt^2 + a(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

- If the gravitational source is spherical symmetric, external fields must be spherical symmetric;
- No matter rest or not, if the distribution inside the gravitational source keeps spherical symmetric, external gravitation fields must stay rest and spherical symmetry;

Suppose functions a and b could be written as

$$a(r) \equiv e^{\mu(r)} \qquad\qquad b(r) \equiv -e^{\nu(r)}$$

after calculating their gauges, Christoffel symbols, Ricci tensors, and inserting them into vacuum field equation

$$\frac{d\mu}{dr} + \frac{d\nu}{dr} = 0 \qquad\qquad 1 - e^{\mu} - r\frac{d\mu}{dr} = 0$$

$$\mu + \nu = A \qquad\qquad e^{-\mu} = 1 - \frac{B}{r}$$

According to Newtonian gravitational potential, we get the constant B as $2GM/c^2$, and finally

$$ds^2 = -c^2\left(1 - \frac{2GM}{c^2r}\right)dt^2 + \left(1 - \frac{2GM}{c^2r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

it is the Schwarzschild solution for field equations, which only depends on the total mass of source. If r turns to infinity:

$$ds^2 = -c^2dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Define gravitational radius,

$$r_g = \frac{2GM}{r^2}$$

namely the event horizon of black holes. Inside the event horizon, r is a time coordinate, instead of a space coordinate. dr is not the proper distance, unless r turns to infinity.

Atomic Emission Spectrometry, its intrinsic frequency changes with the frame of reference, but the number of vibration keeps invariant.

$$\nu_1 d\tau_1 = dN_1 = dN_2 = \nu_2 d\tau_2$$

$$\nu_2 = \sqrt{\left(1 - \frac{2GM}{c^2r_1}\right) / \left(1 - \frac{2GM}{c^2r_2}\right)} \cdot \nu_1$$

in the infinity, we always observe the gravitation redshift, since the clock becomes slower.

The perihelion shift of Mercury's orbit, is the second-order effect of $\left(\frac{GM}{c^r}\right)$, which is the most important one for GR.

$$\Delta\varphi \propto \left(\frac{GM}{c^2R}\right)^2$$

According to GR, we could predict that the light would change the direction inside the gravitational field, compatible with "gravitational redshift". Both are the first-order effect of $\left(\frac{GM}{c^r}\right)$. Strictly, "gravitational redshift" only verifies the equivalent principle, while the other two cases verify the field equation.

8 Black Hole...

Schwarzschild spacetime is the Riemannian spacetime with gauge difference ± 2. In such a spacetime similar to Minkowski one, with undetermined gauge, there would probably be a special hypersurface, whose normal vector is null vector. Since the normal vector is not zero, however with zero length, we call such hypersurface "Null hypersurface", whose normal vector is also the tangent vector at the same time. Suppose

$$f(x^\mu) = f(x^0, x^1, x^2, x^3) = 0$$

is a 3-dim hypersurface in 4-dim spacetime, with normal vector defined as

$$n_\mu = \frac{\partial f}{\partial x^\mu}$$

Define the length of normal vector

$$n_\mu n^\mu = g^{\mu\nu} n_\mu n_\nu = g^{\mu\nu} \frac{\partial f}{\partial x^\mu} \frac{\partial f}{\partial x^\nu}$$

If the eq above is equal to 0, then the hypersurface is null hypersurface. If with symmetry in the spacetime, we call this special null hypersurface the event horizon.

As for the Schwarzschild solution,

$$ds^2 = -c^2 \left(1 - \frac{2GM}{c^2 r}\right) dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

there is a singular point at $r = 0$, and a singular surface for $r = \frac{2GM}{c^2}$ (only coordinate singularity due to non-ideal coordinate system chosen without spacetime curvature divergence here). The latter one is just the surface of the black hole, or the infinite redshift surface, where the clock becomes infinitely slow.

Inside the black hole, since $r < \frac{2GM}{c^2}$, $g_{00} > 0$, iso-r surface inside the hole is indeed isochronous surface, or unidirectional membrane. $r = 0$ is not the sphere center, the end of time instead (not included in spacetime). Similarly, there is white hole, whose sphere center is the beginning of time, with the different direction.

Suppose the material point falls freely from $r = r_0$, and describe with I. D. Novikow coordinate system:

$$ds^2 = -d\tau^2 + \left(\frac{R^2+1}{R^2}\right)\left(\frac{\partial r}{\partial R}\right)^2 dR^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

Time coordinate of Novikow system is just the proper time of the material point above. (τ, R) could be related to (t, r) (Schwarzschild coordinate) by parameters (η, r_0)

$$\begin{cases} \tau = \frac{r_0}{2}\left(\frac{r_0}{2M}\right)^{1/2}(\eta + \sin\eta) \\ R = \left(\frac{r_0}{2M} - 1\right)^{1/2} \end{cases}$$

$$\begin{cases} t = 2M \ln\left|\frac{\left(\frac{r_0}{2M}-1\right)^{1/2} + \tan\frac{\eta}{2}}{\left(\frac{r_0}{2M}-1\right)^{1/2} - \tan\frac{\eta}{2}}\right| + 2M\left(\frac{r_0}{2M}-1\right)^{1/2}\left[\eta + \frac{r_0}{4M}(\eta + \sin\eta)\right] \\ r = \frac{r_0}{2}(1 + \cos\eta) \end{cases}$$

if $\eta = 0$, $r = r_0$ and $\tau = 0$, the material point stays rest. When it arrives at the surface of black hole $r = 2M$, we have

$$\cos\eta = \frac{4M}{r_0} - 1$$

At this moment τ is not infinite. If $r = 0$, namely at the singular point, we have $\eta = \pi$, then

$$\tau = \frac{\pi r_0}{2}\left(\frac{r_0}{2M}\right)^{1/2}$$

Therefore, as for material points, it could penetrate the event horizon and arrive the singular point in finite time. However, for the observer far away, material points could only approach the surface, and furnish more redshift signals, instead of falling in.

Define turtle coordinate

$$r* = r + 2M \ln\left|\frac{r - 2M}{2M}\right|$$

then the event horizon at $r \to 2M$ turns to $r* \to -\infty$.

Define Eddington-Finkelstein coordinates v and u, or Eddington coordinates for short

$$v = t + r \qquad\qquad u = t - r$$

v, advanced Eddington coordinate; u, delayed Eddington coordinate. So the Schwarzschild is presented by u and v shown below

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dv^2 + 2dvdr + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$$

$$ds^2 = -\left(1 - \frac{2M}{r}\right)du^2 - 2dudr + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$$

we might regard u or v as"time", without divergence at singular surface, covering spacetime in or out the event horizon and itself.

Introduce Kruskal coordinate system to cover the whole Schwarzschild metric
$r > 2M$, I zone

$$\begin{cases} T = 4M\left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} sh\frac{t}{4M} \\ R = 4M\left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} ch\frac{t}{4M} \end{cases}$$

$r > 2M$, II zone

$$\begin{cases} T = -4M\left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} sh\frac{t}{4M} \\ R = -4M\left(\frac{r}{2M} - 1\right)^{1/2} e^{r/4M} ch\frac{t}{4M} \end{cases}$$

$r < 2M$, F zone

$$\begin{cases} T = 4M\left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} ch\frac{t}{4M} \\ R = 4M\left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} sh\frac{t}{4M} \end{cases}$$

$r < 2M$, P zone

$$\begin{cases} T = -4M\left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} ch\frac{t}{4M} \\ R = -4M\left(1 - \frac{r}{2M}\right)^{1/2} e^{r/4M} sh\frac{t}{4M} \end{cases}$$

Re-write ds^2 in Schwarzschild spacetime

$$ds^2 = \frac{2M}{r} e^{-r/2M}(-dT^2 + dR^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where (R, T) Kruskal coordinates, T time coordinate, R for space coordinate. r could be presented by R and T:

$$16M^2\left(\frac{r}{2M} - 1\right) e^{r/2M} = R^2 - T^2$$

Kruskal coordinate system could describe the total Schwarzschild spacetime, covering the event horizon of black holes. Also, it could describe all processes (black holes, white holes). All geodesic could be extended into infinity (not including those toward essential singular points). I zone, the universe outside the black hole; F zone, black hole; P zone, white hole; II zone, another universe outside the black hole, without any information communication with ours.

Penrose diagram. By conformal transformation, infinity in Minkowski spacetime would be compressed into finite distance.

In physics and astronomy, the Reissner–Nordström metric is a static solution to the Einstein field equations, which corresponds to the gravitational field of a charged, non-rotating, spherically symmetric body of mass M. The analogous solution for a charged, rotating body is given by the Kerr–Newman metric.

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$$

In 1963, R. P. Kerr obtained a static solution to the Einstein field equations (axial symmetric)

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left[(r^2 + a^2)\sin^2\theta + \frac{2Mra^2\sin^4\theta}{\rho^2}\right] d\phi^2 - \frac{4Mra\sin^2\theta}{\rho} dt d\phi$$

$$\rho^2 = r^2 + a^2\cos^2\theta \qquad \Delta = r^2 - 2Mr + a^2$$

the gauge does not include t and ϕ, so this is static axial-symmetric spacetime. But it does not stay rest, because of the presence of $dtd\phi$.

There are two parameters for this solution, mass M and angular momentum J. ($a = J/M$) Later, E. T. Newman etc found Kerr-Newman solution for charged cases, describing a gravitation field outside one charged rotating star:

$$ds^2 = -\left(1 - \frac{2Mr - Q^2}{\rho^2}\right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2$$
$$+ \left[(r^2 + a^2)\sin^2\theta + \frac{(2Mr - Q^2)a^2 \sin^4\theta}{\rho^2}\right] d\phi^2 - \frac{2(2Mr - Q^2)a\sin^2\theta}{\rho^2} dtd\phi$$

$$\rho^2 = r^2 + a^2\cos^2\theta \qquad \Delta = r^2 - 2Mr + a^2 + Q^2$$

Similar with Kerr solution, this is a static, axial-symmetric spacetime, without t and ϕ in the gauge (but no orthogonal axes). This solution depends on three parameters: the total mass M, the total angular momenta J, the total charge Q.

Note that all solutions above are set in the vacuum, and we say that externals of stars are all in vacuum. If charged, there is only electromagnetic field in the externals.

If $M \neq 0$, $J \neq 0$ but $Q = 0$, Kerr-Newman spacetime returns to Kerr spacetime; if $M \neq 0$, $Q \neq 0$ but $J = 0$, it turns to R-N spacetime; if $M \neq 0$, $J = 0$, $Q = 0$, then to Schwarzschild spacetime.

If $r = 0$, $\theta = \pi/2$, as well as

$$r_{\pm} = \frac{GM}{c^2} \pm \sqrt{\left(\frac{GM}{c^2}\right)^2 - \left(\frac{J}{Mc}\right)^2 - \frac{GQ^2}{c^4}}$$

K-N spacetime has singular gauge.

As for $\nu = \nu_0\sqrt{-g_{00}}$, if redshift is unlimited, we know that $g_{00} = 0$. Then we could obtain a solution with two infinite redshift surfaces for K-N spacetime.

$$r^s_{\pm} = M \pm \sqrt{M^2 - a^2\cos^2\theta - Q^2}$$

The event horizon for Kerr-Newman spacetime ($g^{\mu\nu}\partial_\mu f \partial_\nu f = 0$)

$$r_{\pm} = M \pm \sqrt{M^2 - a^2 - Q^2}$$

Material points could stay rest inside or outside the infinite redshift surface of K-N spacetime. However, it could not at the surface. In the space-like zone, staying rest means superluminal motion, forbidden by GR. Mach thought that a rotating object would drag substances nearby by $\Omega = d\phi/dt$. At the event horizon, $\hat{g}_{00} = 0$, and the dragging velocity is determined by only one value

$$\Omega = \lim_{r \to r_{\pm}} \left(\frac{-g_{03}}{g_{33}}\right) = \frac{a}{r^2_{\pm} + a^2}$$

Define surface gravity κ, the limit of multiplication between its proper acceleration b and redshift factor, while one object approaches the surface of black holes.

$$\kappa_{\pm} = \lim_{r \to r_{\pm}} b\sqrt{-\hat{g}_{00}} = \frac{r_+ - r_-}{2(r^2_{\pm} + a^2)}$$

For charged Kerr-Newman black holes, we could also calculate their electrostatic potential at two poles of the event horizon.

$$V_\pm = \frac{Qr_\pm}{r_\pm^2 + a^2}$$

No-hair theorem: The no-hair theorem states that all black hole solutions of the Einstein–Maxwell equations of gravitation and electromagnetism in general relativity can be completely characterized by only three externally observable classical parameters: mass, electric charge, and angular momentum.

J. Bekenstein & L. Smarr found the equation:

$$M = \frac{\kappa_+}{4\pi}A_+ + 2\Omega_+ J + V_+ Q$$

$$A_+ = 4\pi(r_+^2 + a^2) \qquad \kappa_+ = \frac{r_+ - r_-}{2(r_+^2 + a^2)} \qquad \Omega_+ = \frac{a}{r_+^2 + a^2} \qquad V_+ = \frac{Qr_+}{r_+^2 + a^2}$$

$$dM = \frac{\kappa_+}{8\pi}dA_+ + \Omega_+ J + V_+ Q$$

which is quite similar with First Law of Thermodynamics:

$$dU = TdS + \Omega dJ + VdQ$$

Similarly, there should be the entropy S for black holes, proportional to the area; and temperature T proportional to surface gravity κ.

$$S = \frac{k_B}{4}A_+\left(\frac{c^3}{G\hbar}\right) \qquad\qquad T = \frac{\hbar\kappa_+}{2\pi k_B c}$$

As the extreme black holes, $M^2 = a^2 + Q^2$, inner and outer event horizons united, leading to zero surface gravity. So extreme black holes could be regarded as black holes in absolute zero temperature. According to Third Law of Thermodynamics, we could not make a non-extreme black hole become an extreme black hole by finite operations. To sum up, we have 4 laws of black hole thermodynamics:

- 0: the surface gravity κ for static black holes is a constant;
- 1: $dM = \frac{\kappa_+}{8\pi}dA_+ + \Omega_+ J + V_+ Q$
- 2: $dA_+ \geq 0$, the surface of black holes would not decrease along the time direction;
- 3: could not make a non-extreme black hole become an extreme black hole by finite operations;

9 Introduction of Cosmology

Cosmological principle: the spatial distribution of matter in the universe is homogeneous and isotropic when viewed on a large enough scale, since the forces are expected to act uniformly throughout the universe, and should, therefore, produce no observable irregularities in the large-scale structuring over the course of evolution of the matter field that was initially laid down by the Big Bang.

In order to obtain the static universe model independent of time, we have to insert "repelling effect" into the field equation, to get finit, boundless static universe model.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$$

The constant Λ is called cosmological constant.

In general, we call the case $\Lambda = 0$ as Friedmann model, $\Lambda \neq 0$ as Lemaitre model. Einstein's static model is just a special one.

Three results from observation support the Big Bang model.

1. Hubble's law: galaxies are moving away from the Earth at speeds proportional to their distance; $v = HD$
2. Abundance of helium in the universe is about 25%;
3. Cosmic microwave background, about 2.7K;

Consider a 3-dim hypersphere embedded into 4-dim Euclidean space,

$$x^\mu x^\mu = x^i x^i + x^4 x^4 = \frac{1}{K}$$

it should be a 3-dim space with constant curvature K

$$dx^4 = -\frac{x^i dx^i}{x^4}$$

$$(dx^4)^2 = \frac{(x^i dx^i)^2}{\frac{1}{K} - x^i x^i} = \frac{K(x^i dx^i)^2}{1 - K x^i x^i}$$

$$d\sigma^2 = dx^\mu dx^\mu = dx^i dx^i + \frac{K(x^i dx^i)^2}{1 - K x^i x^i}$$

$$d\sigma^2 = \frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

If $K > 0$, it is the normal hypersphere; if $K = 0$, namely normal hyperplane; if $K < 0$, a pseudo-hypersphere.

From cosmological principle, the curvature of 3-dim space should be all the same everywhere. Even though it probably changes along time, K is not the function of space, only the function of time. Thus the general form for gauges of 4-dim universe:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

this is the important Roberston-Walker gauge, with K . Here, K could be positive, negative, or zero. With follow-up coordinate system (coordinates of material points would not change if expansion), the distance of two points is proportional to a(t), which is called the scale factor.

$$\rho = \frac{3}{8\pi G}\left[\frac{K}{a^2} + \left(\frac{\dot{a}}{a}\right)^2\right] = \frac{3K}{8\pi G a^2} + \frac{3}{8\pi G}H^2$$

Define the critical density

$$\rho_c = \frac{3}{8\pi G}H^2 = 5 \times 10^{-30} g/cm^3 \approx 3 \text{ nucleons}/m^3$$

What's more

$$p = -\frac{1}{8\pi G}\left[\frac{K}{a^2} + H^2(1 - 2q_0)\right]$$

where

$$q_0 = -\frac{\ddot{a}a}{\dot{a}^2}$$

is called deceleration parameter. As for the case $p = 0$, $K = a^2H^2(2q_0 - 1)$.

Density	Deceleration parameter	Curvature	Type	Characteristics
$\rho > \rho_c$	$q_0 > 1/2$	> 0	Finite, boundless	Expansion or Contraction
$\rho = \rho_c$	$q_0 = 1/2$	$= 0$	Infinite, boundless	Expansion
$\rho < \rho_c$	$q_0 < 1/2$	< 0	Infinite, boundless	Expansion

From the obeservation, $\rho < \rho_c$ but $q_0 > 1/2$. Even though it's different from the tableau above, the curvature is almost 0... Also, it's found that the deceleration parameter depends on time, the universe would expand increasingly. To explain, "dark energy" was supposed. There are plenty of dark matter, as well as dark energy with repelling effect inside the universe. The evolution of universe might be a synthetic result by matter, dark matter, and dark energy.